

Analysis of the Magneto-acoustic Tomography with Magnetic Induction (MAT-MI)

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Abstract. Magnetoacoustic tomography with magnetic induction (MAT-MI) is a coupled-physics medical imaging modality for determining conductivity distribution in biological tissue. The capability of MAT-MI to provide high resolution images has been demonstrated experimentally. MAT-MI involves two steps. The first step is a well-posed inverse source problem for acoustic wave equation, which has been well studied in the literature. This paper concerns mathematical analysis of the second step, a quantitative reconstruction of the conductivity from knowledge of the internal data recovered in the first step, using techniques such as time reversal. The problem is modeled by a system derived from Maxwell's equations. We show that a single internal data determines the conductivity. A global Lipschitz type stability estimate is obtained. A numerical approach for recovering the conductivity is proposed and results from computational experiments are presented.

1. Introduction

Electrical conductivity of the biological tissues can provide important information for clinical and research purposes. Conductivity imaging has been a subject of research for decades and the literature is vast.

Magnetoacoustic tomography with magnetic induction (MAT-MI) is a new noninvasive modality for imaging electrical conductivity distribution of biological tissue [18, 11, 13]. In the experiments, the biological tissue is placed in a static magnetic field. A pulsed magnetic field is applied and induces an eddy current inside the conductive tissue. Consequently, the Lorentz force, the force acting on currents in the static magnetic field, causes vibrations and the tissue emits ultrasound waves. The ultrasonic signals are measured around the object. MAT-MI belongs to the class of coupled-physics imaging method which is often referred to as ‘hybrid imaging’. For a review on hybrid imaging methods that recover electrical conductivity distribution, we refer to [17].

Hybrid imaging typically involves two inverse problems. In MAT-MI the two steps are decoupled. The first step involves an inverse source problem for the acoustic wave equation. This problem has been studied extensively in many works including [6, 8, 9, 10, 15, 14]. The second step, the focus of this work, is to reconstruct the spatially varying electrical conductivity from knowledge of the acoustic source.

In the MAT-MI experiment, the object to be imaged is placed in a constant static magnetic background field $\mathbf{B}_0 = (0, 0, 1)$. A pulsed magnetic stimulation is introduced. The pulsed field is of the form $\mathbf{B}_1 u(t)$, where the vector field \mathbf{B}_1 is a constant and $u(t)$ is the time variation. The magnetic permeability of biological tissue is approximately equal to that of a vacuum. Therefore the tissue does not have any noticeable effect on the magnetic field itself. As a result, the time-dependence of the electromagnetic fields is $u(t)$ and we need only to consider their spatial dependence. Because the electric field will depend on conductivity σ , we write it as \mathbf{E}_σ . Let Ω denote the domain to be imaged. Then it can be shown that the electric field satisfies

$$\begin{cases} \nabla \times \mathbf{E}_\sigma &= \mathbf{B}_1, & \text{in } \Omega, \\ \nabla \cdot (\sigma \mathbf{E}_\sigma) &= 0, & \text{in } \Omega, \\ \sigma \mathbf{E}_\sigma \cdot \nu &= 0, & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

The first step in the MAT-MI inverse problem is to recover the acoustic source in the scalar wave equation from observed data at a set of locations. The acoustic source is related to the electromagnetic field; knowledge of the acoustic source in this model is equivalent to knowing the quantity $\nabla \cdot (\sigma \mathbf{E}_\sigma \times \mathbf{B}_0)$ throughout Ω .

In this paper, we focus on the second step of MAT-MI, i.e., reconstruction of the conductivity σ from the internal data given by $\nabla \cdot (\sigma \mathbf{E}_\sigma \times \mathbf{B}_0)$. Our main result is that, if the conductivity is a priori known near the boundary, then it can be uniquely and stably reconstructed from one internal data. More precisely, the main result of this work reads as follows.

Theorem 1.1 *Denote the forward map, the map from conductivity to acoustic source, as $F(\sigma) := \nabla \cdot (\sigma \mathbf{E}_\sigma \times \mathbf{B}_0)$. Suppose that σ_1 and σ_2 satisfy Assumption 2.1 and the support of $\sigma_1 - \sigma_2$ is away from the boundary of Ω at a distance greater than some constant $r_0 > 0$. Then, there exists a constant K , which only depends on r_0 , λ , Λ and Ω , such that, if*

$$\|\nabla \sigma_1\|_{L^\infty} < K, \quad (1.2)$$

then the inequality

$$\|\sigma_1 - \sigma_2\|_{L^2(\Omega)} \leq 4\|F(\sigma_1) - F(\sigma_2)\|_{L^2(\Omega)}, \quad (1.3)$$

holds true.

During the completion of this work, we discovered a recent paper by Ammari, Boulrier and Millien [1]. Their work also focused on the conductivity reconstruction aspect of MAT-MI. What is different is that the authors chose to reconstruct first the current density in the medium. They propose methods to solve for conductivity from current density. In our approach, we directly deal with the relationship between the acoustic source and the electromagnetic field, and propose a method that finds the conductivity from the acoustic source.

The rest of the paper is organized as follows. Section 2 introduces the notation used and basic results needed. In Section 3, we study the mathematical model of the second step of MAT-MI and the linearized version of this problem. Section 4 is devoted to addressing the uniqueness and stability estimate of both linearized and nonlinear problems. In Section 5, we propose an numerical method for solving the inverse problem and present some results from computational experiments. A final section discusses our findings.

2. Notations and preliminaries

We begin by introducing the notations for the the mathematical analysis. Throughout this paper, the standard notations for continuous differentiable function spaces and Sobolev spaces are used. Let Ω be a bounded domain in \mathbb{R}^3 with Lipschitz boundary $\partial\Omega$. A typical point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ denotes the spatial variable. We use the notation $C^\infty(\Omega)$ for infinitely differentiable functions on Ω and $C_0^\infty(\Omega)$ is a subset of $C^\infty(\Omega)$ which contains the functions with compact support. We use $\langle \cdot, \cdot \rangle$ to denote the inner product in the Hilbert space $L^2(\Omega)$. For $p \geq 1$, we denote by $W^{1,p}(\Omega)$ the L^p -based Sobolev spaces on Ω with the usual norm,

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{n=1}^3 \left\| \frac{\partial u}{\partial x_n} \right\|_{L^p(\Omega)}.$$

In the case $p = 2$, we use the notation $H^1(\Omega) = W^{1,2}(\Omega)$, which is a Hilbert space. The Sobolev space $H_0^1(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. The dual space of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$. If there is no danger of confusion, we omit the domain Ω

and abbreviate with L^2 , $W^{1,p}$, H^1 , H_0^1 and H^{-1} . In the following, we do not distinguish in the notation for inner product, function spaces and the corresponding norms between scalar- and vector-valued functions.

Assumption 2.1 *Let σ be a positive function belongs to $W^{1,\infty}$ and assume that*

$$\sigma(x) \geq \lambda, \quad \forall x \in \Omega. \quad (2.1)$$

and

$$\|\sigma\|_{W^{1,\infty}} \leq \Lambda$$

for some constants $\lambda, \Lambda > 0$.

We start with stating several useful results on the elliptic partial differential equations with Neumann boundary condition.

Definition 2.2 *We say that $u \in H^1$ is a weak solution of the Neumann boundary value problem,*

$$\begin{cases} \nabla \cdot (\sigma \nabla u) &= -\nabla \cdot \mathbf{E}, & \text{in } \Omega, \\ (\sigma \nabla u + \mathbf{E}) \cdot \nu &= 0, & \text{on } \partial\Omega, \end{cases} \quad (2.2)$$

if

$$\int_{\Omega} \sigma \nabla u \cdot \nabla \varphi \, dx = - \int_{\Omega} \mathbf{E} \cdot \nabla \varphi \, dx, \quad \forall \varphi \in H^1.$$

We need the following regularity result and standard energy estimate of the gradient.

Proposition 2.1 *Suppose that σ satisfies Assumption 2.1. For field $\mathbf{E} \in L^2$, the Neumann problem (2.2) has a solution $u \in H^1$. The solution u is unique up to an additive constant and satisfies the estimate,*

$$\|\nabla u\|_{L^2} \leq \lambda^{-1} \|\mathbf{E}\|_{L^2}. \quad (2.3)$$

Proof The proof of the existence and uniqueness up to an additive constant is a standard result by the Lax-Milgram Theorem. We refer the readers to [16]. In the following, we prove the gradient estimate (2.3).

It follows from the ellipticity condition (2.1) that

$$\lambda \|\nabla u\|_{L^2}^2 \leq \int_{\Omega} \sigma |\nabla u|^2 \, dx.$$

Taking the test function φ in Definition 2.2 to be the solution u , we have that

$$\int_{\Omega} \sigma \nabla u \cdot \nabla u \, dx = - \int_{\Omega} \mathbf{E} \cdot \nabla u \, dx.$$

Consequently, applying the Cauchy-Schwarz inequality, we obtain that

$$\lambda \|\nabla u\|_{L^2}^2 \leq \left| - \int_{\Omega} \mathbf{E} \cdot \nabla u \, dx \right| \leq \|\nabla u\|_{L^2} \|\mathbf{E}\|_{L^2},$$

and (2.3) follows. \square

3. Analysis of the forward problem

3.1. The forward problem

The second step of MAT-MI is modeled by (1.1), where ν is the unit outer normal vector of $\partial\Omega$ and $\mathbf{B}_1 = (0, 0, 1)$ is a constant vector. The data for this inverse problem is the acoustic source recovered from the first step, namely, $\nabla \cdot (\sigma \mathbf{E}_\sigma \times \mathbf{B}_0)$ with $\mathbf{B}_0 = (0, 0, 1)$. The inverse problem of the second step of MAT-MI consists of reconstruction of conductivity σ from knowledge of $\nabla \cdot (\sigma \mathbf{E}_\sigma \times \mathbf{B}_0)$.

We refer the readers to [3] for the regularity results of the Maxwell's equations. In Proposition 3.1, we show some regularity results of our reduced system (1.1).

Definition 3.1 *We say that $\mathbf{E}_\sigma \in L^2$ is a weak solution of the (1.1) if*

$$\int_{\Omega} \mathbf{E}_\sigma \cdot (\nabla \times \Phi) \, dx = \int_{\Omega} \Phi \cdot \mathbf{B}_1 \, dx, \quad \forall \Phi \in H_0^1,$$

and

$$\int_{\Omega} \sigma \mathbf{E}_\sigma \cdot \nabla \varphi \, dx = 0, \quad \forall \varphi \in H^1.$$

We define the forward problem as

$$\begin{aligned} F : W^{1,\infty} &\rightarrow L^2, \\ \sigma &\mapsto \nabla \cdot (\sigma \mathbf{E}_\sigma \times \mathbf{B}_0). \end{aligned} \tag{3.1}$$

Next, we introduce a proposition on the existence, uniqueness and uniform L^2 -boundedness of the electrical field \mathbf{E}_σ . This implies that forward operator F is well-defined.

Proposition 3.1 *Let σ satisfy Assumption 2.1. Then the system (1.1) is uniquely solvable and there exists a constant C_1 depending on λ , Λ and Ω , such that*

$$\|\mathbf{E}_\sigma\|_{L^2} \leq C_1.$$

Proof This proposition will be derived as a consequence of Proposition 2.1. Let us first reduce the system (1.1) to a Neumann boundary problem. Let $\tilde{\mathbf{E}} = \frac{1}{2}(-y, x, 0)$. We can readily check that $\nabla \times \tilde{\mathbf{E}} = \mathbf{B}_1$. Hence $\nabla \times (\mathbf{E}_\sigma - \tilde{\mathbf{E}}) = 0$ and we can write $\mathbf{E}_\sigma = \tilde{\mathbf{E}} + \nabla u$. Substituting this into (1.1), we have that u solves the Neumann boundary problem,

$$\begin{cases} \nabla \cdot (\sigma \nabla u) &= -\nabla \cdot (\sigma \tilde{\mathbf{E}}), & \text{in } \Omega, \\ (\sigma \nabla u + \sigma \tilde{\mathbf{E}}) \cdot \nu &= 0, & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

The existence of u and uniqueness of ∇u follows from Proposition 2.1. For the uniqueness of \mathbf{E}_σ , we consider the equations

$$\begin{cases} \nabla \cdot (\sigma \nabla v) &= 0, & \text{in } \Omega, \\ \sigma \nabla v \cdot \nu &= 0, & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

If both \mathbf{E}_1 and \mathbf{E}_2 are solutions to the system (1.1), then we have that $\mathbf{E}_1 - \mathbf{E}_2 = \nabla v$ and v solves the equations (3.3). By Proposition 2.1, the only H^1 solutions to (3.3) are constants. Hence ∇v vanishes and \mathbf{E}_σ is unique.

What remains is to show the L^2 boundedness of \mathbf{E}_σ . Applying Proposition 2.1 to u , we have that

$$\|\nabla u\|_{L^2} \leq \lambda^{-1} \|\sigma \tilde{\mathbf{E}}\|_{L^2}.$$

Hence,

$$\|\mathbf{E}_\sigma\|_{L^2} = \|\tilde{\mathbf{E}} + \nabla u\|_{L^2} \leq (\Lambda/\lambda + 1) \|\tilde{\mathbf{E}}\|_{L^2}.$$

Note that we can choose $\tilde{\mathbf{E}} = \frac{1}{2}(-y + a, x + b, 0)$ and repeat the above argument for any constants a and b . It follows that,

$$\|\mathbf{E}_\sigma\|_{L^2} \leq C_1,$$

where

$$C_1 = \frac{1}{2}(\Lambda/\lambda + 1) \inf_{a,b} \|(-y + a, x + b, 0)\|_{L^2},$$

only depends on λ, Λ and Ω . □

3.2. Linearization of the forward map

Recall that the distribution of the electric field \mathbf{E}_σ depends nonlinearly on the conductivity σ and $\nabla \cdot (\sigma \mathbf{E}_\sigma \times \mathbf{B}_0)$ is nonlinear with respect to σ . It is natural to start by linearizing the relationship between conductivity and data. In this section, we introduce the linearized of the inverse problem. We first examine the Fréchet differentiability of the forward operator F . Then, some useful properties of the Fréchet derivative at σ , DF_σ , are presented.

To introduce the Fréchet derivative, we consider the following Neumann boundary problem,

$$\begin{cases} \nabla \cdot (\sigma \nabla \varphi_h) &= -\nabla \cdot (h \mathbf{E}_\sigma), & \text{in } \Omega, \\ (\sigma \nabla \varphi_h + h \mathbf{E}_\sigma) \cdot \nu &= 0, & \text{on } \partial\Omega, \end{cases} \quad (3.4)$$

where $h \in W^{1,\infty}$ is the increment to the conductivity.

Theorem 3.2 *For σ satisfying Assumption 2.1, the forward operator F , defined in (3.1), is bounded and Fréchet differentiable at σ . Its Fréchet derivative at σ , DF_σ , is given by*

$$DF_\sigma(h) = \nabla \cdot ((\sigma \nabla \varphi_h + h \mathbf{E}_\sigma) \times \mathbf{B}_0), \quad (3.5)$$

where φ_h solves (3.4), and satisfies

$$\|DF_\sigma(h)\|_{L^2} \leq C_2 \|h\|_{W^{1,\infty}}, \quad \forall h \in W^{1,\infty}, \quad (3.6)$$

for some constant C_2 depends on λ, Λ and Ω .

Proof We first prove the boundedness of F . We can write

$$\begin{aligned} F(\sigma) &= \nabla \cdot (\sigma \mathbf{E}_\sigma \times \mathbf{B}_0) \\ &= \sigma \nabla \cdot (\mathbf{E}_\sigma \times \mathbf{B}_0) + \nabla \sigma \cdot (\mathbf{E}_\sigma \times \mathbf{B}_0) = \sigma + \nabla \sigma \cdot (\mathbf{E}_\sigma \times \mathbf{B}_0). \end{aligned}$$

It follows, by boundedness of σ and Proposition 3.1, that

$$\|F(\sigma)\|_{L^2} \leq \|\sigma\|_{L^2} + \|\nabla \sigma\|_{L^\infty} \|\mathbf{E}_\sigma\|_{L^2} \leq (|\Omega|^{1/2} + C_1) \|\sigma\|_{W^{1,\infty}},$$

where C_1 is the same constant as in Proposition 3.1.

Next, we show the Fréchet differentiability of F at σ . Consider the data

$$F(\sigma + h) = \nabla \cdot ((\sigma + h) \mathbf{E}_{\sigma+h} \times \mathbf{B}_0)$$

for some $h \in W^{1,\infty}$ such that $\sigma + h$ also satisfies Assumption 2.1, where $\mathbf{E}_{\sigma+h}$ is the solution to (1.1) with σ replaced by $\sigma + h$. Note that

$$\nabla \times (\mathbf{E}_{\sigma+h} - \mathbf{E}_\sigma) = 0.$$

Hence we can write $\mathbf{E}_{\sigma+h} - \mathbf{E}_\sigma = \nabla u$. Substituting this into the equations for $\mathbf{E}_{\sigma+h}$ and \mathbf{E}_σ , we obtain that u solves

$$\begin{cases} \nabla \cdot (\sigma \nabla u) &= -\nabla \cdot (h \mathbf{E}_{\sigma+h}), & \text{in } \Omega, \\ (\sigma \nabla u + h \mathbf{E}_{\sigma+h}) \cdot \nu &= 0, & \text{on } \partial\Omega. \end{cases} \quad (3.7)$$

Applying Proposition 2.1 to u , we have

$$\|\nabla u\|_{L^2} \leq \lambda^{-1} \|h \mathbf{E}_{\sigma+h}\|_{L^2}. \quad (3.8)$$

Let $v = u - \varphi_h$, where φ_h solves (3.4). Then, v solves

$$\begin{cases} \nabla \cdot (\sigma \nabla v) &= -\nabla \cdot (h \nabla u), & \text{in } \Omega, \\ (\sigma \nabla v + h \nabla u) \cdot \nu &= 0, & \text{on } \partial\Omega. \end{cases} \quad (3.9)$$

Applying Proposition 2.1 to v , we have

$$\|\nabla v\|_{L^2} \leq \lambda^{-1} \|h \nabla u\|_{L^2}. \quad (3.10)$$

To estimate the remainder terms, we write

$$\begin{aligned} &F(\sigma + h) - F(\sigma) - \nabla \cdot ((\sigma \nabla \varphi_h + h \mathbf{E}_\sigma) \times \mathbf{B}_0) \\ &= \nabla \cdot ((\sigma (\mathbf{E}_{\sigma+h} - \mathbf{E}_\sigma - \nabla \varphi_h) + h (\mathbf{E}_{\sigma+h} - \mathbf{E}_\sigma)) \times \mathbf{B}_0) \\ &= \nabla \cdot ((\sigma \nabla v + h \nabla u) \times \mathbf{B}_0) \\ &= \nabla \sigma \cdot (\nabla v \times \mathbf{B}_0) + \nabla h \cdot (\nabla u \times \mathbf{B}_0). \end{aligned}$$

Therefore, by (3.8), (3.10) and Proposition 3.1, we have

$$\begin{aligned} &\|F(\sigma + h) - F(\sigma) - \nabla \cdot ((\sigma \nabla \varphi_h + h \mathbf{E}_\sigma) \times \mathbf{B}_0)\|_{L^2} \\ &= \|\nabla \sigma \cdot (\nabla v \times \mathbf{B}_0) + \nabla h \cdot (\nabla u \times \mathbf{B}_0)\|_{L^2} \\ &\leq \|\nabla \sigma\|_{L^\infty} \|\nabla v\|_{L^2} + \|\nabla h\|_{L^\infty} \|\nabla u\|_{L^2} \\ &\leq C_1 \Lambda \lambda^{-2} \|h\|_{L^\infty}^2 + C_1 \lambda^{-1} \|\nabla h\|_{L^\infty} \|h\|_{L^\infty}. \end{aligned}$$

We can readily check the linearity of the operator maps h to $\nabla \cdot ((\sigma \nabla \varphi_h + h \mathbf{E}_\sigma) \times \mathbf{B}_0)$. This complete the proof of Fréchet differentiability of F at σ .

What remains is to show that the formal Fréchet derivative DF_σ is a bounded linear operator. Note that

$$\begin{aligned} DF_\sigma(h) &= \nabla \cdot ((\sigma \nabla \varphi_h + h \mathbf{E}_\sigma) \times \mathbf{B}_0) \\ &= \sigma \nabla \cdot (\nabla \varphi_h \times \mathbf{B}_0) + \nabla \sigma \cdot (\nabla \varphi_h \times \mathbf{B}_0) + h \nabla \cdot (\mathbf{E}_\sigma \times \mathbf{B}_0) + \nabla h \cdot (\mathbf{E}_\sigma \times \mathbf{B}_0) \\ &= h + \nabla \sigma \cdot (\nabla \varphi_h \times \mathbf{B}_0) + \nabla h \cdot (\mathbf{E}_\sigma \times \mathbf{B}_0). \end{aligned}$$

By applying Proposition 2.1 to φ_h and Proposition 3.1 to \mathbf{E}_σ , we conclude that

$$\begin{aligned} &\|DF_\sigma(h)\|_{L^2} \\ &= \|h + \nabla \sigma \cdot (\nabla \varphi_h \times \mathbf{B}_0) + \nabla h \cdot (\mathbf{E}_\sigma \times \mathbf{B}_0)\|_{L^2} \\ &\leq \|h\|_{L^2} + \Lambda \|\nabla \varphi_h\|_{L^2} + \|\nabla h\|_{L^\infty} \|\mathbf{E}_\sigma\|_{L^2} \\ &\leq |\Omega|^{1/2} \|h\|_{L^\infty} + \Lambda \lambda^{-1} \|h \mathbf{E}_\sigma\|_{L^2} + C_1 \|\nabla h\|_{L^\infty} \\ &\leq (|\Omega|^{1/2} + C_1(\Lambda \lambda^{-1} + 1)) \|h\|_{W^{1,\infty}}. \end{aligned}$$

□

4. Uniqueness and stability

In the following theorem, we obtain a Lipschitz type stability estimate for the inverse problem under certain conditions on the conductivity. The uniqueness of the inverse problem follows.

Theorem 4.1 *Suppose that σ satisfy Assumption 2.1. If σ only depends on the third component of the spatial variable, i.e. $\sigma(x) = \sigma(x_3)$, then the inequality*

$$\|DF_\sigma(h)\|_{L^2(\Omega)} \geq \frac{1}{2} \|h\|_{L^2(\Omega)} \quad (4.1)$$

holds true for any $h \in W_0^{1,\infty}(\Omega)$.

Proof Note that

$$\nabla \cdot (\nabla \varphi_h \times \mathbf{B}_0) = 0,$$

for any H^2 function φ_h and that

$$\nabla \sigma \times \mathbf{B}_0 = (0, 0, \frac{\partial \sigma}{\partial x_3}) \times (0, 0, 1) = 0.$$

Hence,

$$\nabla \cdot (\sigma \nabla \varphi_h \times \mathbf{B}_0) = \sigma \nabla \cdot (\nabla \varphi_h \times \mathbf{B}_0) + \nabla \sigma \cdot (\nabla \varphi_h \times \mathbf{B}_0) = 0.$$

Therefore,

$$DF_\sigma(h) = \nabla \cdot ((\sigma \nabla \varphi_h + h \mathbf{E}_\sigma) \times \mathbf{B}_0) = \nabla \cdot (h \mathbf{E}_\sigma \times \mathbf{B}_0).$$

Multiplying the both sides by h and integrating over Ω , we obtain that

$$\int_{\Omega} h DF_\sigma(h) \, dx = \int_{\Omega} h \nabla \cdot (h \mathbf{E}_\sigma \times \mathbf{B}_0) \, dx.$$

By using the integration by parts twice, we have

$$\begin{aligned}
 \int_{\Omega} h D F_{\sigma}(h) \, dx &= \int_{\Omega} h \nabla \cdot (h \mathbf{E}_{\sigma} \times \mathbf{B}_0) \, dx \\
 &= - \int_{\Omega} (h \mathbf{E}_{\sigma} \times \mathbf{B}_0) \cdot \nabla h \, dx \\
 &= - \frac{1}{2} \int_{\Omega} (\mathbf{E}_{\sigma} \times \mathbf{B}_0) \cdot \nabla (h^2) \, dx \\
 &= \frac{1}{2} \int_{\Omega} h^2 \nabla \cdot (\mathbf{E}_{\sigma} \times \mathbf{B}_0) \, dx \\
 &= \frac{1}{2} \|h\|_{L^2(\Omega)}^2.
 \end{aligned}$$

The last identity above follows by noting

$$\nabla \cdot (\mathbf{E}_{\sigma} \times \mathbf{B}_0) = \nabla \times \mathbf{E}_{\sigma} \cdot \mathbf{B}_0 = \mathbf{B}_1 \cdot \mathbf{B}_0 = 1.$$

Then, by applying Cauchy-Schwarz inequality to $\int_{\Omega} h D F_{\sigma}(h) \, dx$, we obtain (4.1). \square

The same technique can be used to provide a general stability estimate for the nonlinear inverse problem. Note that, in the following theorem and corollary, no smallness constraint on the difference of conductivities is needed.

Theorem 4.2 *Suppose that σ_1 and σ_2 satisfy Assumption 2.1. If $\sigma_1 - \sigma_2 \in W_0^{1,\infty}(\Omega)$ and*

$$\nabla \sigma_1 \times \nabla \sigma_2 \cdot \mathbf{B}_0 = 0, \tag{4.2}$$

then the inequality

$$\|\sigma_1 - \sigma_2\|_{L^2(\Omega)} \leq 2 \|F(\sigma_1) - F(\sigma_2)\|_{L^2(\Omega)}, \tag{4.3}$$

holds true.

Proof Assume that E_1 and E_2 solve (1.1) with σ replaced by σ_1 and σ_2 , respectively. Let us multiply $F(\sigma_1) - F(\sigma_2)$ by $\sigma_1 - \sigma_2$ and integrate over Ω to obtain

$$\begin{aligned}
 &\int_{\Omega} (\sigma_1 - \sigma_2)(F(\sigma_1) - F(\sigma_2)) \, dx \\
 &= \int_{\Omega} (\sigma_1 - \sigma_2)(\nabla \cdot (\sigma_1 \mathbf{E}_1 - \sigma_2 \mathbf{E}_2) \times \mathbf{B}_0) \, dx \\
 &= \int_{\Omega} (\sigma_1 - \sigma_2)(\nabla \cdot ((\sigma_1 - \sigma_2) \mathbf{E}_1 \times \mathbf{B}_0) + \nabla \cdot (\sigma_2(\mathbf{E}_1 - \mathbf{E}_2) \times \mathbf{B}_0)) \, dx \\
 &= \frac{1}{2} \|\sigma_1 - \sigma_2\|_{L^2(\Omega)}^2 + \int_{\Omega} (\sigma_1 - \sigma_2) \nabla \cdot (\sigma_2(\mathbf{E}_1 - \mathbf{E}_2) \times \mathbf{B}_0) \, dx.
 \end{aligned} \tag{4.4}$$

In the above inequalities, the last step follows by the similar argument as in the proof of Theorem 4.1.

Next, we estimate

$$\int_{\Omega} (\sigma_1 - \sigma_2) \nabla \cdot (\sigma_2(\mathbf{E}_1 - \mathbf{E}_2) \times \mathbf{B}_0) \, dx.$$

Recall that $\nabla \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$. Hence, we can write $\mathbf{E}_1 - \mathbf{E}_2 = \nabla u$. Applying integration by parts twice, we obtain that

$$\begin{aligned}
 & \int_{\Omega} (\sigma_1 - \sigma_2) \nabla \cdot (\sigma_2 (\mathbf{E}_1 - \mathbf{E}_2) \times \mathbf{B}_0) \, dx \\
 &= - \int_{\Omega} \sigma_2 \nabla (\sigma_1 - \sigma_2) \cdot (\nabla u \times \mathbf{B}_0) \, dx \\
 &= \int_{\Omega} \sigma_2 (\nabla (\sigma_1 - \sigma_2) \times \mathbf{B}_0) \cdot \nabla u \, dx \\
 &= - \int_{\Omega} \nabla \cdot (\sigma_2 (\nabla (\sigma_1 - \sigma_2) \times \mathbf{B}_0)) u \, dx \\
 &= - \int_{\Omega} (\sigma_2 \nabla \cdot (\nabla (\sigma_1 - \sigma_2) \times \mathbf{B}_0) + \nabla \sigma_2 \times \nabla (\sigma_1 - \sigma_2) \cdot \mathbf{B}_0) u \, dx \\
 &= 0.
 \end{aligned} \tag{4.5}$$

Here we use the equalities (4.2),

$$\nabla \cdot (\nabla (\sigma_1 - \sigma_2) \times \mathbf{B}_0) = 0,$$

and

$$\nabla \sigma_2 \times \nabla \sigma_2 = 0.$$

Combining (4.4) and (4.5), we discover

$$\int_{\Omega} (\sigma_1 - \sigma_2) (F(\sigma_1) - F(\sigma_2)) \, dx = \frac{1}{2} \|\sigma_1 - \sigma_2\|_{L^2(\Omega)}^2.$$

The stability estimate (4.3) follows by applying the Cauchy-Schwarz inequality to the left-hand side of the above equality. \square

In the following corollary, we list some simple cases, in which, the criteria (4.2) is easy to check.

Corollary 4.1 *Suppose that σ_1 and σ_2 satisfy Assumption 2.1. If $\sigma_1 - \sigma_2 \in W_0^{1,\infty}(\Omega)$ and satisfy any one of the following three conditions:*

- (i) σ_1 only depends on the third component of the spatial variable x_3 ;
- (ii) There exists a real number t such that $t\sigma_1 + (1-t)\sigma_2$ only depends on x_3 ;
- (iii) There exist a positive integer N and real numbers a_n , $n = 1, 2, \dots, N$ such that $\sigma_2 + \sum_{n=1}^N a_n (\sigma_1 - \sigma_2)^n$ only depends on x_3 ;

then the stability estimate (4.3) holds true.

Proof We can readily see that (i) and (ii) are simple cases of (iii). It suffices to show that (4.2) is satisfied and apply Theorem 4.2.

From (iii), we know that

$$\nabla \left(\sigma_2 + \sum_{n=1}^N a_n (\sigma_1 - \sigma_2)^n \right) \times \mathbf{B}_0 = \mathbf{0}. \tag{4.6}$$

In light of (4.6) and the facts that

$$\nabla \sigma_2 \times \nabla \sigma_2 = \mathbf{0},$$

we have the following equalities,

$$\begin{aligned} & \nabla \sigma_1 \times \nabla \sigma_2 \cdot \mathbf{B}_0 \\ &= \nabla(\sigma_1 - \sigma_2) \times \nabla \sigma_2 \cdot \mathbf{B}_0 \\ &= \nabla(\sigma_1 - \sigma_2) \times \nabla \left(\sigma_2 - \left(\sigma_2 + \sum_{n=1}^N a_n (\sigma_1 - \sigma_2)^n \right) \right) \cdot \mathbf{B}_0 \\ &= \nabla(\sigma_1 - \sigma_2) \times \left(\sum_{n=1}^N a_n n (\sigma_1 - \sigma_2)^{n-1} \nabla(\sigma_1 - \sigma_2) \right) \cdot \mathbf{B}_0 \\ &= \left(\sum_{n=1}^N a_n n (\sigma_1 - \sigma_2)^{n-1} \nabla(\sigma_1 - \sigma_2) \times \nabla(\sigma_1 - \sigma_2) \right) \cdot \mathbf{B}_0 \\ &= 0. \end{aligned}$$

The proof is completed by applying Theorem 4.2. \square

Roughly speaking, in Theorem 4.2, we prove that, if the structure of two conductivities satisfies the condition (4.2), the inverse problem bears a Lipschitz stability estimate. We propose next to remove this structure condition. In Theorem 1.1, we show that, if one conductivity varies less dramatically, the Lipschitz type stability estimates also holds true.

Proof of Theorem 1.1 The proof differs from the one of Theorem 4.2 in the treatment of the last term in (4.4),

$$I \triangleq \int_{\Omega} (\sigma_1 - \sigma_2) \nabla \cdot (\sigma_2 (\mathbf{E}_1 - \mathbf{E}_2) \times \mathbf{B}_0) \, dx.$$

We continue from (4.4). First, we estimate the electric field difference. Note that $\mathbf{E}_1 - \mathbf{E}_2$ is curl-free and we set

$$\nabla u = \mathbf{E}_1 - \mathbf{E}_2.$$

Then, u satisfies the equation

$$\begin{cases} \nabla \cdot (\sigma_1 \nabla u) = -\nabla \cdot ((\sigma_1 - \sigma_2) \mathbf{E}_2), & \text{in } \Omega, \\ \nabla u \cdot \nu = 0, & \text{on } \partial\Omega. \end{cases} \quad (4.7)$$

Applying Proposition 2.1 to u , we obtain that

$$\|\nabla u\|_{L^2} \leq \lambda^{-1} \|(\sigma_1 - \sigma_2) \mathbf{E}_2\|_{L^2}.$$

From the standard L^p estimate of elliptic equations [7, Chapter 9] and the Sobolev Embedding Theorem, we know that \mathbf{E}_2 is bounded and

$$\|\mathbf{E}_2\|_{L^\infty} < C,$$

where C only depends on r_0 , λ , Λ and Ω . Thus, we conclude that

$$\|\mathbf{E}_1 - \mathbf{E}_2\|_{L^2} \leq C \|(\sigma_1 - \sigma_2)\|_{L^2}.$$

Now, with the choice of K such that $KC \leq 1/4$, we estimate $|I|$ as follows:

$$\begin{aligned}
 |I| &= \left| \int_{\Omega} (\sigma_1 - \sigma_2) \nabla \cdot (\sigma_2 (\mathbf{E}_1 - \mathbf{E}_2) \times \mathbf{B}_0) dx \right| \\
 &= \left| \int_{\Omega} (\sigma_1 - \sigma_2) \nabla \sigma_2 \cdot ((\mathbf{E}_1 - \mathbf{E}_2) \times \mathbf{B}_0) dx \right| \\
 &\leq \|\nabla \sigma_2\|_{L^\infty} \|\sigma_1 - \sigma_2\|_{L^2} \|\mathbf{E}_1 - \mathbf{E}_2\|_{L^2} \\
 &\leq C \|\nabla \sigma_2\|_{L^\infty} \|\sigma_1 - \sigma_2\|_{L^2}^2 \\
 &\leq \frac{1}{4} \|\sigma_1 - \sigma_2\|_{L^2}^2.
 \end{aligned} \tag{4.8}$$

Substituting (4.8) into (4.4), we discover that

$$\int_{\Omega} (\sigma_1 - \sigma_2) (F(\sigma_1) - F(\sigma_2)) dx \geq \frac{1}{4} \|\sigma_1 - \sigma_2\|_{L^2(\Omega)}^2.$$

The desired estimate (1.3) follows by applying the Cauchy-Schwarz inequality to the left-hand side. \square

5. An iterative reconstruction scheme

One possible approach to solving the inverse problem is to formulate it as a least-squares problem. One can then apply a gradient-based method to solve the least-squares problem. Such a method will require knowledge of the Fréchet derivative of the forward map which we studied in Section 3. Convergence analysis of this type of reconstruction approach is available in [4, 5]. Results in these references, together with our analysis of DF in Sections 3 and 4 can be used to provide a convergence analysis for the iterative reconstruction of MAT-MI using steepest descent method. The main challenge of the least-squares approach lies in the difficulty to accurately evaluate DF and its adjoint where numerical differentiations are involved. We temporarily abandon the least-squares approach in favor of one that is based on a fixed point method. This approach is described next.

5.1. Formulation

In view of the structure of this inverse problem, we propose a novel iterative scheme, in which, the forward map and its derivative are not required. The desired conductivity is updated by solving a stationary advection-diffusion equation. Let σ^\dagger denote the unknown conductivity to be reconstructed, \mathbf{E}^\dagger be the corresponding electric field and g be the internal data obtained in the first step of MAT-MI. The internal data is related to the conductivity and the field through

$$g = \nabla \cdot (\sigma^\dagger \mathbf{E}^\dagger \times \mathbf{B}_0).$$

The algorithm proceeds as follows:

(S0) Select an initial conductivity σ_0 and set $k = 0$;

(S1) Calculate the associated electric field \mathbf{E}_k by solving the boundary value problem

$$\begin{cases} \nabla \times \mathbf{E}_k &= \mathbf{B}_1, & \text{in } \Omega, \\ \nabla \cdot (\sigma_k \mathbf{E}_k) &= 0, & \text{in } \Omega, \\ \sigma_k \mathbf{E}_k \cdot \nu &= 0, & \text{on } \partial\Omega. \end{cases} \quad (5.1)$$

(S2) Calculate the updated conductivity by solving the stationary advection-diffusion equation:

$$\begin{cases} \nabla \cdot (\sigma_{k+1} \mathbf{E}_k \times \mathbf{B}_0) &= g, & \text{in } \Omega, \\ \sigma_{k+1} &= \sigma_0, & \text{on } \partial\Omega. \end{cases} \quad (5.2)$$

(S3) Set $k = k + 1$ and go to (S1).

Convergence test can be based on $\|\sigma_k - \sigma_{k-1}\|$ or based on data misfit $\|g - \nabla \cdot (\sigma_{k+1} \mathbf{E}_{k+1} \times \mathbf{B}_0)\|$.

5.2. Convergence analysis

The main advantage of this scheme is two-fold: First, the update of the conductivity is calculated directly using the the measured data and the simulated electric field. Hence, fewer numerical differentiations are involved when compared to the gradient-based least-squares minimization. Second, the convergence analysis can be carried out using an idea similar to the one in the proof of Theorem 1.1. A global convergence result and a linear convergence rate are established the following theorem.

Theorem 5.1 *Suppose that the true conductivity σ^\dagger satisfies Assumption 2.1 and*

$$\|\nabla \sigma^\dagger\|_{L^\infty} < 2K, \quad (5.3)$$

where the constant K is the same as in Theorem 1.1, which only depends on r_0 , λ , Λ and Ω . Then, for any initial σ_0 satisfying Assumption 2.1 and coinciding with σ^\dagger over the boundary $\partial\Omega$, the above algorithm generates a sequence $\{\sigma_k\}$, $k = 0, 1, \dots$, which is convergent to σ^\dagger and satisfies

$$\|\sigma_k - \sigma^\dagger\|_{L^2} \leq c^k \|\sigma_0 - \sigma^\dagger\|_{L^2}, \quad k = 0, 1, \dots, \quad (5.4)$$

where $c < 1$ depends on $\|\nabla \sigma^\dagger\|_{L^\infty}$ and Ω .

Proof We start by subtracting $\nabla \cdot (\sigma^\dagger \mathbf{E}_k \times \mathbf{B}_0)$ from both sides of (5.2) to obtain

$$\nabla \cdot ((\sigma_{k+1} - \sigma^\dagger) \mathbf{E}_k \times \mathbf{B}_0) = \nabla \cdot (\sigma^\dagger (\mathbf{E}^\dagger - \mathbf{E}_k) \times \mathbf{B}_0).$$

Multiplying the both sides by $\sigma_{k+1} - \sigma^\dagger$ and integrating over Ω , we arrive at

$$\begin{aligned} & \frac{1}{2} \|\sigma_{k+1} - \sigma^\dagger\|_{L^2}^2 \\ &= \int_{\Omega} (\sigma_{k+1} - \sigma^\dagger) \nabla \cdot ((\sigma_{k+1} - \sigma^\dagger) \mathbf{E}_k \times \mathbf{B}_0) dx \\ &= \int_{\Omega} (\sigma_{k+1} - \sigma^\dagger) \nabla \cdot (\sigma^\dagger (\mathbf{E}^\dagger - \mathbf{E}_k) \times \mathbf{B}_0) dx \\ &= \int_{\Omega} (\sigma_{k+1} - \sigma^\dagger) \nabla \sigma^\dagger \cdot ((\mathbf{E}^\dagger - \mathbf{E}_k) \times \mathbf{B}_0) dx. \end{aligned}$$

In the above identities, the first identity follows from a similar argument to the one used in the proof of Theorem 4.1 and the last identity follows by noting that $(\mathbf{E}^\dagger - \mathbf{E}_k) \times \mathbf{B}_0$ is divergence-free. Next, we estimate the electric field difference. As in the proof of Theorem 1.1, we conclude that

$$\|\mathbf{E}^\dagger - \mathbf{E}_k\|_{L^2} \leq C\|\sigma_k - \sigma^\dagger\|_{L^2}.$$

By the Cauchy-Schwarz inequality and (5.3), we have that

$$\|\sigma_{k+1} - \sigma^\dagger\|_{L^2} \leq c\|\sigma_k - \sigma^\dagger\|_{L^2}$$

and (5.4) follows from an induction argument on k . \square

Remark 5.2 *Let us point out that indeed the convergence analysis of the proposed algorithm carries through when the inverse problem have a Lipschitz type stability estimate. In fact, Theorem 1.1 still holds true with the condition (1.2) replaced by (5.3). Correspondingly, the stability constant will depend on $\|\nabla\sigma^\dagger\|_{L^\infty}$ and blow up as $\|\nabla\sigma^\dagger\|_{L^\infty}$ approaches $2K$.*

5.3. Numerical experiments

Now we present some numerical experiments to verify the convergence theory presented in the previous subsection. For each experiment, the true conductivity is assumed to be Lipschitz continuous and equal to 0.2 near the boundary and we use constant 0.2 as the initial model unless otherwise specified. To simplify the computation, we transform the 3D problem into a 2D problem by assuming the conductivity is invariant along the x_3 direction. The setup is as follows. The domain we take is the square $\Omega = (0, 1) \times (0, 1)$. We employ a uniform triangulation with a mesh size of $1/64$. Both the Neumann problem and the stationary advection-diffusion equation are solved using a first-order finite element method. The algorithm is implemented using FEniCS, a finite element software package [12], and using Python as the user interface. All the numerical computations are performed on a dual-core laptop computer.

Example 1. We first consider a simple example. The true conductivity is shown in Figure 1a and the error between the true and reconstructed model is shown in Figure 1b. The relative L^2 -error, $\|\sigma_k - \sigma^\dagger\|/\|\sigma^\dagger\|$, drops to 2.88×10^{-7} after 16 iterations. As shown in Figure 1c, a linear convergence rate is observed.

Example 2. We then attempt to recover a more complicated conductivity model, as shown in Figure 2a. In Figure 2b, we show the absolute error of the recovered model after 45 iterations. The relative L^2 -error drops to 2.57×10^{-7} and a linear convergence rate is still observed. In this example, the gradient of the conductivity is greater than the one in the previous example. According to Theorem 5.1, this will lead to a greater prefactor c in the convergence rate. The comparison of Figure 1c and Figure 2c demonstrates this point.

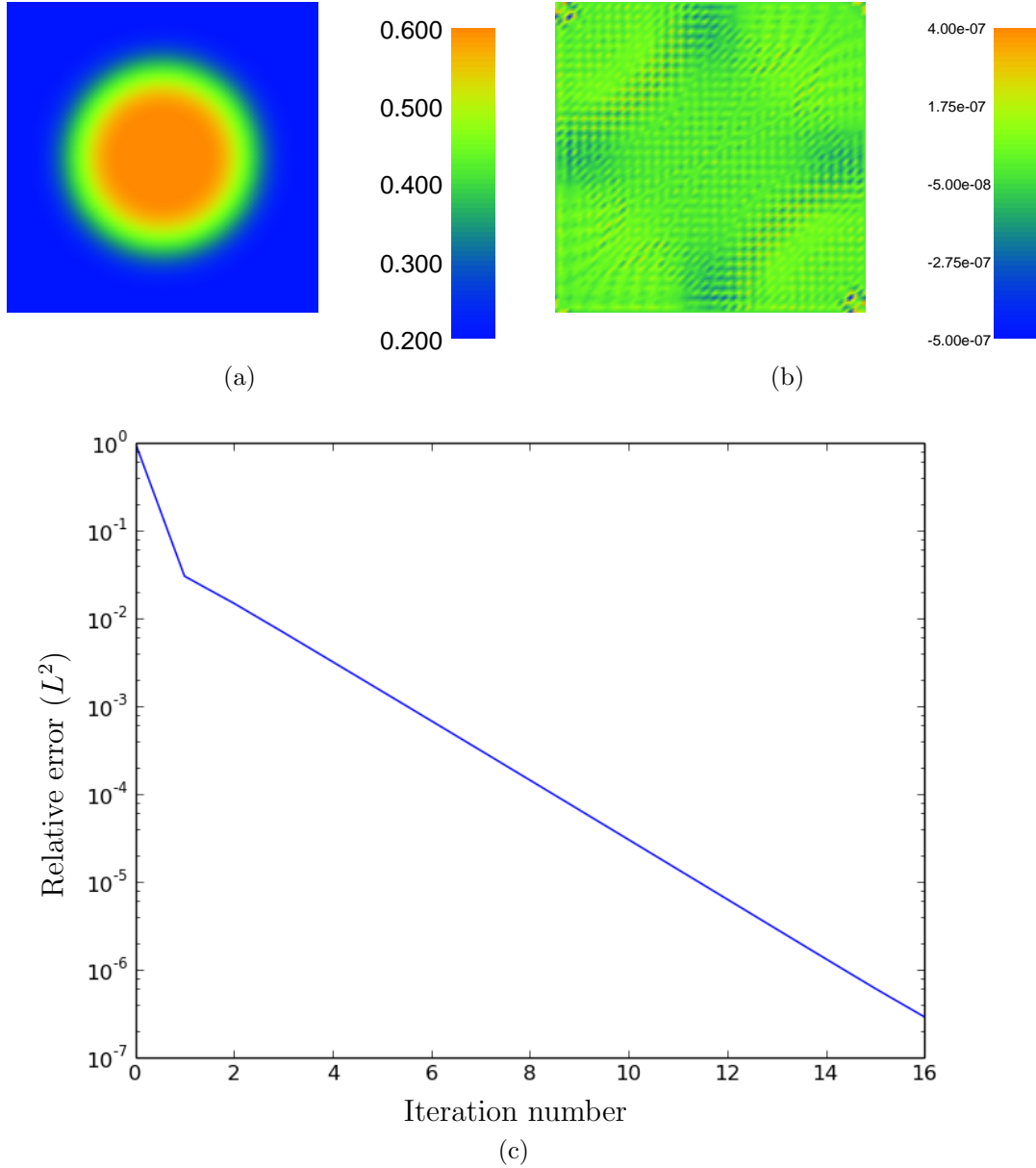


Figure 1: (a) true conductivity model, (b) absolute error between the inverted and true conductivity model, (c) decay of the relative error in L^2 norm (logarithmic scale).

Example 3. To further investigate and demonstrate the convergence results in Theorem 5.1, we perform the third test, which is the “reverse” Example 2. We switch the role of the true model and the initial model in Example 2. That is, we try to recover the constant conductivity with an initial model as shown in Figure 2a. The algorithm converges after 1 iteration with the absolute L^2 -error drops below 5×10^{-8} . This implies that the prefactor c approaches zero as the true conductivity goes to a constant function. Actually, this can be proved by noticing that, when g is constant, the unique solution to (5.2) is the same constant for any admissible σ_0 .

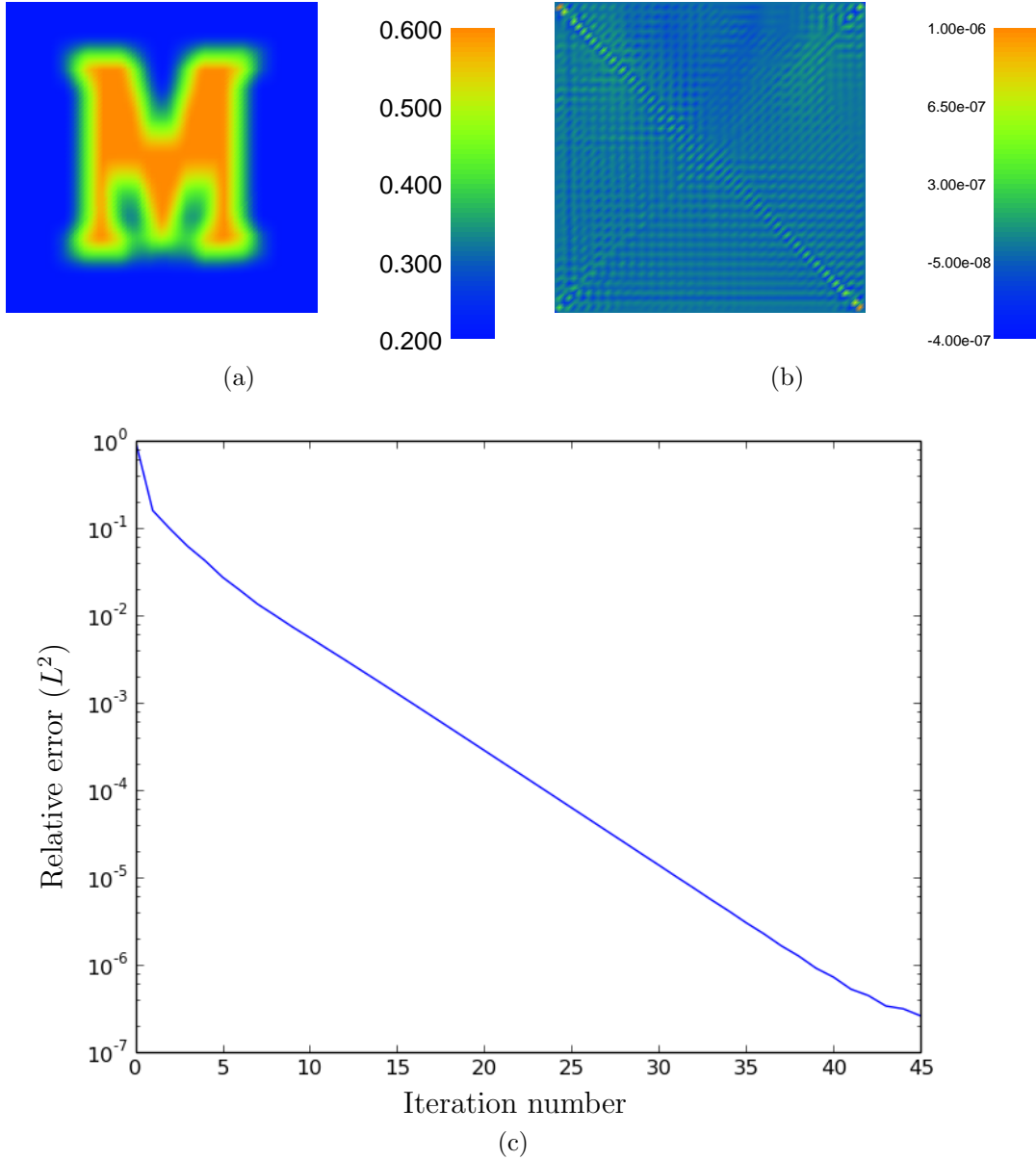


Figure 2: (a) true conductivity model, (b) absolute error between the inverted and true conductivity model, (c) decay of the relative error in L^2 norm (logarithmic scale).

6. Discussion

We investigated the second step in MAT-MI where the problem is to reconstruct the conductivity distribution from internal data obtained in the first step. A global Lipschitz type stability estimate is established when the conductivity is $W^{1,\infty}$. We devise a novel iterative method for solving the inverse problem that involves, at each iteration, the solution of a well-posed boundary value problem followed by the solution of an advection-diffusion problem. The iterative method is shown to be convergent. Results from numerical experiments demonstrate the effectiveness of the approach.

It would be interesting to extend the computational method proposed to three dimensions and to invert real measured data. An important direction for this research is to consider the case of anisotropic conductivity. In [2], the authors examine the effect of electrical anisotropy in MAT-MI. A homogeneous tissue is considered. They find that, when imaging nerve or muscle, electrical anisotropy has a significant effect on the acoustic signal and must be accounted for in order to obtain accurate images.

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